Euclidean Geometry" in *The Popular Science Monthly* for November, 1905: "Among conditions to a more profound understanding of even very elementary parts of the euclidean geometry, the knowledge of the non-euclidean geometry can not be dispensed with."

And now finally of the tremendous achievement to which all this is but a prelude, Sommerville's Bibliography of Non-Euclidean Geometry, suffice it to say that it is an immortal monument, showing to all what we of the cult have ever realized, that here we face one of the very few germinal ideas whose creation has made our modern world.

John Bolyai published but two dozen pages; what I have long called the most extraordinary two dozen pages in the whole history of thought. Here we see shown us a list of their fruit more than four thousand articles and books dealing with this subject.

The following is a bibliography of Sommerville's productions:

1905. "Semi-regular networks of the plane in absolute geometry." Edinburgh, Trans. R. Soc., 41, 725-747 (12 plates); Edinburgh, Proc. R. Soc., 25, 392-394 [abstract].

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1906. "On the distribution of the proper fractions." Edinburgh, Proc. R. Soc., 26, 116-129.

"On the classification of frequency-ratios." Biometrika, Cambridge, 5, 179-181.

"Questions 15894, 15903, 15906, 15948, 15925." Math. Questions, *Educational Times*, London, 10, 34, 40, 71, 83-84, 93-94.

1907. "On links and knots in euclidean space of n dimensions." *Messenger Math.*, Cambridge, 36, 139-144.

"On certain projective configurations in space of n dimensions and a related problem in arrangements." Edinburgh, Proc. Math. Soc., 25, 80-90.

"Questions 15986, 15999, 16009." Math. Questions, *Educational Times*, London, 11, 24-25, 55-56, 57-61. [Q. 16009, proposed by H. Bateman, "On systems of hyperspheres, each touching two others in succession." Generalized for euclidean or non-euclidean space of any dimensions.]

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"Question 15951." Math. Questions, *Educational Times*, London, 12, 24-25.

1909. "On certain periodic properties of cyclic compositions of nnmbers." London, Proc. Math. Soc., (2) 7, 263-313.

1910. "A problem in voting." Edinburgh, Proc. Math. Soc., 28.
"Classification of geometries with projective metric." Edinburgh, Proc. Math. Soc., 28, 25-41.
"Elementary considerations relating to the absolute." Edinburgh, Proc. Math. Soc., 28, 65-72.
"Note on the geometries in which straight lines are represented by circles." Edinburgh, Proc. Math. Soc., 28, 81-94.
"The early history of non-euclidean geometry." *Nature*, London, 84, 172.
"On the need of a non-euclidean bibliography." Rep. Brit. Assoc., 80 (Sheffield), 531-532.
1911. "The numerically greatest term of a binomial expansion." Edinburgh, Math. Notes, No. 7, 74-77.
"Concrete representations of non-euclidean geometry." In a Collection of Scientific Papers published by the University of St. Andrews to commemorate the 500th anniversary of its founda-
"Bibliography of non-euclidean geometry, including the theory of parallels, the foundations of geometry, and space of n dimensions." London: Harrison, pp. 400.

NOTES ON GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE OF INTEGERS.

By BENJAMIN FRANKLIN YANNEY, Wooster University.

I. Properties of Quotients.

Let $q_1, q_2, \ldots, q_n$ be the respective quotients obtained by dividing the positive integers $a_1, a_2, \ldots, a_n$ by their greatest common divisor D; and $q'_1, q'_2, \ldots, q'^n$ the respective quotients obtained by dividing L, the lowest common multiple of the integers $a_1, a_2, \ldots, a_n$, by the integers, in turn. Hence

$$q_1 = a_1 \div D;\ q_2 = a_2 \div D;\ \ldots;\ q_n = a_n \div D \ldots (1);$$
$$q'_1 = L \div a_1;\ q'_2 = L \div a_2;\ \ldots;\ q'_n = L \div a_n \ldots (2).$$

The two sets of quotients possess the following properties:

1. Neither set of quotients has a factor common to all the numbers in the set. For, otherwise, D would not be the greatest common divisor nor L the least common multiple of the numbers $a_1, a_2, \ldots, a_n$.

2. $q_1 q'_1 = q_2 q'_2 = \ldots = q_n q'_n = L \div D$.

This result is obtained by multiplying, member by member, the cor-

responding equalities in (1) and (2). Furthermore, by suitably combining all the equalities in (1) and (2), we get

3. $(q_1q_2\ldots q_n)(q'_1q'_2\ldots q'_n)=L^n\div D^n$; and,

4. $(a_1a_2\ldots a_n)^2=L^nD^n(q_1q_2\ldots q_n)\div(q'_1q'_2\ldots q'_n)$.

From the second property stated above it is easily seen that

5. $L\div D$ divided in order by either set of quotients gives in corresponding order the other set.

From the first and fifth properties, it follows that

6. The least common multiple of each set of quotients is $L\div D$.

II. Relation of Integers to their Greatest Common Divisor and Least Common Multiple.

1. As is well known, the product of two positive integers, a_1 and a_2, is equal to the product of their greatest common divisor D and least common multiple L:

$$a_1a_2=DL.$$

The general relation may be stated as follows:

$$DL^{n-1}\overline{\overline{>}}\, a_1a_2\ldots a_n\overline{\overline{>}}\, D^{n-1}L\ldots(\text{A}).$$

Proof. From (2) of I, we get

$$L^n=(a_1a_2\ldots a_n)(q'_1q'_2\ldots q'_n).$$

But, as previously shown, the least common multiple of $q'_1, q'_2, \ldots, q'_n$ is $L\div D$. Hence, $q'_1q'_2\ldots q'_n\overline{\overline{>}}\, L\div D$. Therefore, $L^n\overline{\overline{>}}\,(a_1a_2\ldots a_n)L\div D$.

$\therefore DL^{n-1}\overline{\overline{>}}\, a_1a_2\ldots a_n$.

Similarly, starting with (1) of I, it can be shown that

$$a_1a_2\ldots a_n\overline{\overline{>}}\, D^{n-1}L.$$

2. Another form of the general case is as follows:

$$\prod {}_kD_i \prod {}_kL_i^{k-1}\overline{\overline{>}}\,(a_1a_2\ldots a_n)^{\frac{(n-1)!}{(n-k)!(k-1)!}}\overline{\overline{>}}\,\prod {}_kD_i^{k-1}\prod {}_kL_i,\ldots(\text{B}),$$

in which ${}_kD_i$ and ${}_kL_i$ represent, respectively, the greatest common divisors and least common multiples of the n integers taken k at a time, i thus being

equal to 1, 2, ..., $\frac{n!}{(n-k)!k!}$.

The proof of this relation follows easily from (A), which is true for each set of k integers selected from the entire set of n integers. There will evidently be, in all, $\frac{n!}{(n-k)!k!}$ such relations. Multiplying together these relations, member by member, and noticing that in the middle member each of the n integers will occur $\frac{(n-1)!}{(n-k)!(k-1)!}$ times, the desired result is obtained.

3. If in (B) we set $k=2$, we obtain

$$(a_1 a_2 \ldots a_n)^{n-1} = \prod_2 D_i \prod_2 L_i, \quad i=1, 2, \ldots, \tfrac{1}{2}n(n-1),$$

which is probably the most general equality relation existing between integers, and greatest common divisors and least common multiples.

If, finally, $n=2$, we arrive at the well-known relation

$$a_1 a_2 = DL.$$

A POINT'S VISIT TO THE LINEAR CONTINUUM.

By H. W. REDDICK, Columbia University.

Once upon a time a point, XYZ, in three-dimensional space, decided to visit the set of points on a certain straight line. He had a sense of his superiority over his less fortunate fellow-beings who were compelled to lead a one-dimensional existence, but his motiqe for visiting them might be called philanthropic for he had a real desire to find out the relations existing between them and to enlighten them, if possible, concerning a higher existence.

As XYZ approached the line he was surprised to find that its residents were not living happily together; and he soon learned that they were capable of the same petty dissensions and jealousies which he had supposed were possible only in the enlightened set to which he belonged. He found that none of them was on speaking terms with his next-door neighbor — in fact, that he did not know who his next-door neighbor was. The society was divided up into political parties, religious denominations, and exclusiqe cliques, which spent most of their time quarrelling with one another, and did not seem to realize that, taken all together, they formed one Grand Continuum.

The society was divided into two great political sets, the Rationalists and the Irrationalists. The members of the Rational Set believed that the tariff on any commodity should be expressible in a finite number of figures. Their opponents could see no reason for such a belief. The chief argument of the Rationalists against the Irrationalists was that they were irrational. It was impossible, however, to settle any question by voting because there was always trouble in counting the Irrational vote.

There were two great religious denominations, the Transcendental and the Algebraic sects. The Transcendentalists looked upon those of the opposite faith with a feeling of pity, not unmingled with contempt, reproaching them for allowing themselves to be counted and for serving as roots of mere algebraic equations with rational coefficients. But the members of the Algebraic Set were firm in their pragmatic belief and denounced their opponents as being unpractical. Every inhabitant of the Continuum felt that he had the right of membership in one of these religious denominations but there was a great multitude who did not know to which set they should belong. However, there were two saints among the Transcendental Set whose right to be termed orthodox members of the faith had been established beyond doubt. These were e and π. Of these, e was the patron saint, who had been canonized by a human, Hermite, in 1873. e had become distinguished, however, long before this, by being appointed to act as base for the Napierian system of logarithms. But the members of the Transcendental Set never mentioned this distinction when eugolizing e in the presence of a member of the Algebraic Set, for the latter would promptly remind them that one of his number, 10, had achieved a greater glory by holding the office of base of the more practical Briggs system of logarithms. The other saint, π, had come up through great tribulation. Many were the mathematical sins that had been committed in his name! Owing to his pecaliar attribute of being the ratio of a circle to its diameter, he had often been misunderstood and cursed as a member of the Algebraic Set. But his vindication, glorious and complete, was brought about in 1882 by Lindemann. *XYZ* found that the two following questions were the ones most discussed by the Transcendentalists: Which of our number will be the next to be proved worthy of membership? Will a rule ever be found that can be applied to any individual to determine to which set he should belong?

Mr. *XYZ* then turned his attention to some of the more exclusive cliques that had recently been formed. There were several sets, regarded as snobbish by the others, who called themselves perfect, but they were all closed, so that *XYZ* failed to obtain much information about them. One of the most famous of these was Cantor's Typical Ternary Club, a very aristocratic organization whose members looked upon all non-members as the masses who were everywhere dense. The latter, however, referred sneeringly to the members of the organization as of content zero.

Mr. *XYZ* then dropped in at the Integral Club, where a number of in-

tegers had assembled. One of them informed *XYZ* that their members had been added together and had multiplied rapidly and in some instances one had been subtracted from another or a family had been divided, but that in no case had a number been produced which did not belong to the set. "However, we are a discreet set," he added with a smile. "We are careful not to let the root-extractor enter our midst." The members then began discussing their figures, the figures by which they were recognized. "I maintain that I have the most peculiar shape," said 6, "See! when I stand on my head I become 9." "That 's nothing, said 8, "When I lie down I become infinite." "But then you are no longer one of us," replied the others, "What shall we do with him for perpetrating such a joke? We 'll leave it to you, Mr. *XYZ*." "Well," said *XYZ*, "I think he should be killed. Let him that is greatest among you cast the first stone." They were all silent. Then Zero rolled along and tried to start an argument. "You do n't amount to anything anyhow," the others protested. "Just the same," retorted Zero, "I claim a distinction of which none of you can boast: a distinguished human, Mr. Russell, has dedicated a chapter to me."

At this stage Mr. *XYZ*, regarding his mission as hopeless, moved away into space, more firmly grounded in his conviction that the Continuum will never be well-ordered.

AN EXAMPLE OF THE USEFULNESS OF FOURIER'S THEOREM IN SEPARATING THE ROOTS OF EQUATIONS.

By L. R. MANLOVE, England.

Cases must occur in practice where the roots of an equation cannot be separated by any of the well-known easier methods and where Sturm's Theorem is inapplicable by reason of the labor which it involves. In such cases resort may be had to a combination of Fourier's Theorem with Lagrange's method of approximation, as shown below.

EXAMPLE.

Let us take the equation

$$x^{17}-35x^{15}+11x^{14}-1000x^{10}+2500x^{5}-151x^{3}+1=0.$$

A first application of Fourier's Theorem shows that there are:

(a) *Two positive roots:* One between 1 and 2; one between 5 and 6.

(b) *Three negative roots:* One between 0 and -1; one between -1 and -2; one between -6 and -7.

(c) *A doubtful interval* between 0 and +1 in which 4 changes of sign are lost and which *may* consequently include 4 more positive roots.

To dispose of the doubtful interval, put $x=\frac{1}{u_1}$.

The equation in u_1 is:

$$u_1^{17}-151u_1^{14}+2500u_1^{12}-1000u_1^{7}+11u_1^{3}-35u_1^{2}+1=0=(\text{say})\,F_A(u_1).$$

Fourier's functions of $F_A(u_1)$ give for the value 1 of u_1 two changes of sign, and for the value 2 no changes of sign.

$F_A(u_1)=0$ may therefore have two roots between 1 and 2, but has no other positive roots greater than unity.

The four originally doubtful roots are reduced to two.

To dispose of the remaining pair put $u_1=1+\frac{1}{u_2}$.

The equation in u_2 is:

$$1327u_2^{17}+20866u_2^{16}+(9\text{ positive terms})-12588u_2^{6}-5053u_2^{5}$$
$$+(5\text{ positive terms})=0=(\text{say})\;F_B(u_2),$$

which clearly can have no positive root greater than unity.

The four originally doubtful roots are all imaginary and the proposed equation is disposed of.

REMARKS.

The formation (by Horner's method) of the equation $F_B(u_2)=0$ is effected *currente calamo* and the whole example might probably be worked in less than an hour.

It will be observed that in the auxiliary equations used in this method we are only concerned with positive values of the variable greater than unity.

Negative roots of the original equation are most conveniently dealt with by substituting $-y$ for x in that equation and seeking the positive roots of the resulting equation in y.

The writer has tested the method by a number of trials and finds that for most intervals the doubtful roots are at or before the second auxiliary equation, either separated or shown to be imaginary, and in no case, so far, has he found it necessary to go beyond the third auxiliary equation.

Failure to attain definite results within the limits just specified creates a presumption of the existence of equal roots which must be dealt with in the ordinary way before proceeding farther.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

358. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.

Show that $\frac{n(n+1)\ldots(n+m-1)}{m!} - n\frac{n(n+1)\ldots(n+m-4)}{(m-3)!} + \frac{n(n-1)}{2!}\cdot\frac{n(n+1)\ldots(n+m-7)}{(m-6)!} - \ldots = 0$, when $m>2n$; and $=1$ when $m=2n$.

Solution by B. F. FINKEL, Ph. D., Drury College.

We have $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \ldots$

$$+ \frac{n(n+1)\ldots(n+m-1)}{m!}x^m + \ldots$$

Also, $(1-x^3)^n = 1 - nx^3 + \frac{n(n-1)}{1.2}x^6 - \frac{n(n-1)(n-2)}{1.2.3}x^9 +$

$$\ldots + (-1)^m \frac{n(n-1)(n-2)\ldots(n+m-1)}{m!}x^{3m} + \ldots$$

The series under consideration is the coefficient of x^m in the product of the right-hand members of these two equations, which is the same as the coefficient of x^m in the product of $(1-x)^{-n}(1-x^3)^n$ or $(1+x+x^2)^n$.

Now the coefficient of x^m in the expansion of $(1+x+x^2)^n$, by the *multinomial theorem*, is

$$\Sigma \frac{n!}{\alpha!\,\beta!\,\gamma!}, \text{ where } \alpha+\beta+\gamma = n \text{ and } \beta+2\gamma = m.$$

The solutions of these two equations are

$$\gamma = 0, 1, 2, 3, \ldots, n;$$
$$\beta = m, m-2, m-4, m-6, \ldots, m-2n;$$
$$\alpha = n-m, n-m+1, n-m+2, n-m+3, \ldots, 2n-m.$$

$$\therefore \frac{n(n+1)(n+2)....(n+m-1)}{m!} - n\frac{n(n+1)...(n+m-4)}{(m-3)!}$$

$$+\frac{n(n-1)}{1.2.3}\frac{n(n+1)...(n+m-7)}{(m-6)!}...$$

$$=\frac{n!}{(n-m)!\,m!}+\frac{n!}{(n-m+1)!\,(m-2)!\,1!}+\frac{n!}{(n-m+2)!\,(m-4)!\,2!}+...$$

$$+\frac{n!}{(2n-m)!\,(m-2n)!\,n!}.$$

Now, if $m>2n$, there are no terms in the expansion of $(1+x+x^2)^n$ of which the exponent of x is m. Hence, the second member of the above equation is zero. When $m=2n$, there exists only one term of which the exponent of x is $m=2n$, and its coefficient is the last term of the above equation, which is equal to 1.

The fact that there are no terms of which the exponent of x is m when $m>2n$ is manifested also by some of the signs in the factors of the denominators becoming negative.

359. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Illinois.

Show when $1/(1-x)(1-x^3)(1-x^5)(1-x^7)...=(1+x)(1+x^2)(1+x^3)(1+x^4)...$

Solution by B. F. FINKEL, Ph. D., Drury College.

$$\frac{1}{(1-x)(1-x^3)(1-x^5)...(1-x^{2n-1})}$$

$$\equiv\frac{(1+x)(1+x^2)(1+x^3)...(1+x^n)}{(1-x)(1+x)(1+x^2)(1-x^3)(1+x^3)(1+x^4)...(1-x^{2n-1})}$$

$$\equiv\frac{(1+x)(1+x^2)(1+x^3)...(1+x^n)}{(1-x^{2^p-1})(1-x^{3.2^q-1})(1-x^{5.2^r-1})(1-x^{7.2^s-1}).....(1-x^{2n-1})},$$

where $p+q+r+s+...=4n-4$,

$$\doteq(1+x)(1+x^2)(1+x^3)...(1+x^n)... \text{ as } n\doteq\infty,$$

when $-1<x<1$, since, when $-1<x<1$, each factor in the denominator approaches 1.

NOTE. The editors desire that contributors send in good problems for solution. Let us have a great variety of problems for solutions in the various departments. Also send in solutions; we prefer to publish solutions prepared by contributors, rather than publish our own. We have neither time, inclination, nor ability to solve every problem proposed in the MONTHLY. However, if every contributor will give a little time to the problems each month, by united effort, there will be few problems remain unsolved. We have recently republished a number of unsolved problems, and we shall be pleased to have solutions of any of them, or any others that remain unsolved. ED. F.

GEOMETRY.

384. Proposed by S. LEFSHETZ, Universityof Nebraska.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

Solution by R. P. BAKER, Iowa City, Iowa.

I. By elementary geometry.

Lemma I. The second tangent from a', $a'K$ to the circle is parallel to BC.

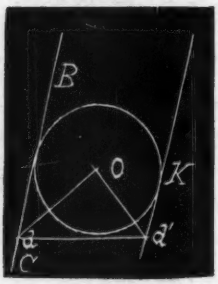

Fig. 1.

For $a'aB$, $aa'A$ are the doubles of the complementary angles $a'aO$ and $aa'O$, and hence supplementary. So $a'K$ is parallel to aB, that is to CB.

Lemma II. If the tangents to a circle from P meet two parallel tangents in Q, Q'; R, R', respectively, and O is the center; then $PQ.PR=OP^2=PQ'.PR'$.

Let M, N be the points of contact of PQ, PR, and S, T of the parallel tangents.

Then $OPQ=OPR$; $2OQP=$supp. $SOM=$supp. $(SOP-MOP)=$supp. $SOP+MOP$.

$2POR=2POT-TON=2POT-(POT-NOP)=POT+MOP=$supp. $SOP+MOP$. Therefore, $OQP=POR$, and the triangles OPQ and RPO are similar.

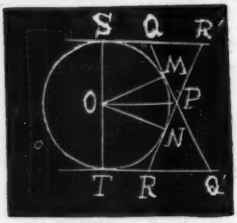

Fig. 2.

Hence $PQ.PR=OP^2$, and by the similar triangles PRQ', $PR'Q$, each is equal to $PQ'.PR'$.

Applying the lemmas to the figure, we have $ab'.aC=ac'.aB$, and the triangles aBb', aCc' are similar, having equal angles at a, and Bb', Cc' are parallel. So for Aa', Bb'.

II. By Brianchon's Theorem.

If the second tangents from b', c' meet at A', we have in AB, BC, CA, $b'A'$, bc, $c'A'$ six tangents to a conic. If the lines are taken in the order written the joins of the cuts of opposite pairs are Bb', Cc', and the line at infinity. Hence Bb', Cc' are parallel, and similarly, the other pairs.

III. Consider the locus of intersections of Bb' and Cc. The pencils at B and C are projective, being in 1 : 1 correspondence with the tangents T by a ruler construction. The intersection is at infinity on AB when T coincides with AB, and at infinity on AC when T coincides with AC. It is on BC only if T coincides with BC. From the latter fact the locus is a straight line, and from the two former the line at infinity, the pencils being in perspective.

IV. The generalized problem.

AB, BC, CA, Ta, OI, OJ are six tangents to a conic: Ta cuts AB in c, BC in a, CA in b; OI, OJ have chord of contact IJ:Oa' is the harmonic conjugate of Oa with respect to OI and OJ:a' is its cut with Ta, and so for Ob', Oc'. Then $Aa'Bb'Cc'$ are concurrent on IJ.

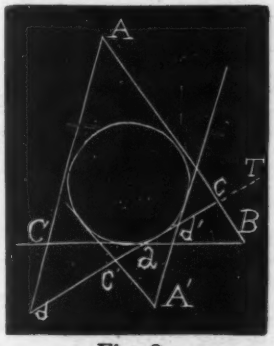

Fig. 3.

Let AB have contact at γ, and OI at κ, IJ at P, OJ at λ. Let PG be the second tangent from P, G its point of contact. $G\gamma$, the polar of P, passes through O, the pole of PIJ. Let it cut IJ at π. Then $PI\pi J$ is harmonic, and so is $P\kappa\gamma\lambda$. The latter is the range determined on the tangent AB by the four tangents PG, OI, AB, OJ. The range on ab being equal, PG cuts this line in c'.

This proves Lemma I in general. Brianchon's Theorem is then applied as in II.

V. By Analytic Geometry.

Take the equation of the circle as

$$x=\frac{1-\lambda^2}{1+\lambda^2};\quad y=\frac{2\lambda}{1+\lambda^2}.$$

The tangent at λ is

$$(1-\lambda^2)x+2\lambda y=1+\lambda^2$$

and intersects the tangent at κ in

$$x=\frac{1-\kappa\lambda}{1+\kappa\lambda};\quad y=\frac{\kappa+\lambda}{1+\kappa\lambda}.$$

The line through the center perpendicular to the join of this point and the center is:

$$x(1-\kappa\lambda)+y(\kappa+\lambda)=0,$$

and cuts the κ tangent in

$$x=\frac{\kappa+\lambda}{\lambda-\kappa};\quad y=\frac{\kappa\lambda-1}{\lambda-\kappa}.$$

The line joining this to the cut of tangents at $(\mu,\ \nu)$ has the slope

$$\frac{\kappa(\lambda+\mu+\nu+\lambda\mu\nu)-(1+\lambda\mu+\mu\nu+\nu\lambda)}{2(\kappa+\lambda\mu\nu)}$$

which is symmetrical in $(\lambda,\ \mu,\ \nu)$. The three lines given by interchanges of $\lambda,\ \mu,\ \nu$ are therefore parallel.

CALCULUS.

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312. Proposed by C. N. SCHMALL, New York City.

Given $y^3-3y+x=0$, prove by Maclaurin's theorem, that

$$y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\text{etc.}$$

I. Solution by H. PRIME.

Put $u=y^3-3y+x=0$. Then $\partial u/\partial x=1$, $\partial u/\partial y=3y^2-3$; hence, $dy/dx=\frac{1}{3}(1-y^2)=(1+y^2+y^4+y^6+\text{etc.})/3$.

$$d^2y/dx^2=(2y+4y^3+6y^5+\text{etc.})(1+y^2+y^4+\text{etc.})/3^2$$
$$=(2y+6y^3+12y^5+\text{etc.})/3^2.$$

$$d^3y/dx^3=(2+18y^2+60y^4+\text{etc.})(1+y^2+y^4+\text{etc.})/3^3$$
$$=(2+20y^2+80y^4+\text{etc.})/3^3.$$

$$d^4y/dx^4=(40y+320y^3+\text{etc.})(1+y^2+y^4+\text{etc.})/3^4$$
$$=(40y+360y^3+\text{etc.})/3^4.$$

$$d^5y/dx^5=(40+1080y^2+\text{etc.})(1+y^2+y^4+\text{etc.})/3^5$$
$$=(40+1120y+\text{etc.})/3^5;\ \text{etc.}$$

When $x=0$, $(y)=0$. Hence

$(dy/dx)=\frac{1}{3}$, $(d^2y/dx^2)=0$, $(d^3y/dx^3)=2/3^3$, $(d^4y/dx^4)=0$, $(d^5y/dx^5)=40/3^5$, etc. By Maclaurin's formula, $y=x/3+x^3/3^4+x^5/3^6+\text{etc.}$

II. Solution by A. M. HARDING, University of Arkansas, and JAMES A. BULLARD, Worcester Polytechnic Institute.

Let $f(x, y) \equiv y^3 - 3y + x = 0$. Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{1}{3y^2-3} = f'(x, y); \quad \frac{d^2y}{dx^2} = \frac{-6y}{(3y^2-3)^3} = f''(x, y);$$

$$\frac{d^3y}{dx^3} = \frac{-6(15y^2+3)}{(3y^2-3)^5} = f'''(x, y); \quad \frac{d^4y}{dx^4} = \frac{-24.45y(2y^2+1)}{(3y^2-3)^7} = f^{iv}(x, y);$$

$$\frac{d^5y}{dx^5} = \frac{-120.9(56y^4+57y^2+3)}{(3y^2-3)^9} = f^{v}(x, y); \ldots$$

Substituting the values of these derivatives for $x=0$, *i. e.* when $y=0$, in Maclaurin's formula, we have

$$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \frac{f^{v}(0)}{5!}x^5 + \ldots$$

$$= \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \ldots$$

NOTE. When $x=0$, $y=0$, $\sqrt{3}$, or $-\sqrt{3}$. Then there are two other series for y obtained by substituting $y=\sqrt{3}$ and $y=-\sqrt{3}$ in the above derivations.

Solved similarly by V. M. Spunar, S. G. Barton, M. E. Graber, and C. N. Schmall.

313. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Evaluate the definite integral $\int_0^\infty (e^{-2ax^2} + e^{2ax^2})\,dx$.

Solution by C. N. SCHMALL, New York City.

$$\int_0^\infty (e^{-2ax^2} + e^{2ax^2})\,dx = \int_0^\infty e^{-2ax^2}\,dx + \int_0^\infty e^{2ax^2}\,dx.$$

The first of these two integrals can be evaluated by meams of the familiar result $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$ (see Williamson's *Integral Calculus*).

Putting $x\sqrt{(2a)}$ for x, we have

$$\int_0^\infty e^{-2ax^2}\,dx = \tfrac{1}{2}\sqrt{\frac{\pi}{2a}}.$$

Now, the expression e^{2ax^2} can be integrated only, so far as we know, by expanding it in an infinite series, thus:

$$e^{2ax^2}=1+2ax^2+\frac{4a^2x^4}{1.2}+\ldots.$$

$$\therefore \int e^{2ax^2}dx=x+\frac{2ax^3}{1.3}+\frac{4a^2x^5}{1.2.5}+\ldots.$$

NOTE. Professors Harding and Prime should have received credit for solving 308.

PROBLEMS FOR SOLUTION.

ALGEBRA.

366. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Eliminate m from the equations

$$3a^2m^4+4aym^3+6axm^2-(x^2+y^2-4ax)=0;$$
$$(x^2+y^2-4ax)m^4-6axm^2+4aym-3a^2=0.$$

367. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the simultaneous equations:

$$\frac{2x}{1+x^2}=y\ldots(1);\quad \frac{2y}{1+y^2}=z\ldots(2);\quad \frac{2z}{1+z^2}=u\ldots(3);\quad \frac{2u}{1+u^2}=x\ldots(4).$$

368. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the functional equation, $\frac{f(-x)}{f(x)}=r^{2x}$.

$f(x)=a_1r^x+\sum_{n=2}^{\infty} a_n[r^{nx}+r^{-(n-2)x}]$

GEOMETRY.

399. Proposed by J. K. ELLWOOD, Superintendent of Schools, Lucas, Kansas.

A race track is to be composed of two tangents and the arc of the circle which is concave towards the point of intersection of the two tangents, each tangent and the arc of the circle being 1 mile. What is the radius of the circle?

400. Proposed by FRANCIS RUST, C. E., Pittsburgh, Pennsylvania.

Given a circle and a point P without; construct, *using the straight edge only*, the two tangents to the circle through P.

CALCULUS.

320. Proposed by J. F. LAWRENCE, Stillwater, Oklahoma.

Show that, if $u=1+A_1x+\frac{1}{2!}A_2x^2+\frac{1}{3!}A_3x^3+\ldots$ where the quantities A are connected by the relation $A_m=mA_{m-1}-\frac{1}{2}(m-1)(m-2)A_{m-3}$, then $\log[u(1-x)^{\frac{1}{2}}]=\frac{1}{2}x+\frac{1}{4}x^2$. [From Forsyth's *Differential Equations*, p. 48.]

321. Proposed by ARTEMAS MARTIN, Ph. D., LL. D., United States Coast and Geodetic Survey Office, Washington, D. C.

To a person in a boat at the center of a circular pond the bottom appears to be perfectly level. What is the actual form of the bottom of the pond, the depth of the water at the center being a feet, and the distance of the eyes of the observer from the surface of the water being b feet. [From the *Mathematical Visitor*, Vol. 2, No. 2, p. 62.]

MECHANICS.

267. Proposed by PROFESSOR G. H. LIGHT, Purdue University, Lafayette, Indiana.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests; required the position of equilibrium. [From Bowser's *Mechanics*, Ex. 37, p. 96.]

268. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

ABC and ADE are two uniform congruent wedges, each of weight w. $B=D=90°$. At B and D they are smoothly hinged to a horizontal table. The bases AB, AC all but meet at A, the common foot of two rough inclined faces AC, AE. A rod, weight W, length $2l$, reclines horizontally and symmetrically with an end on each inclined face. Find the conditions of equilibrium.

NOTES AND NEWS.

The winter meeting of the Chicago section of the American Mathematical Society was held in Chicago on Friday and Saturday, December 28, 29, 1911. There were fifty-seven in attendance upon the various sessions, including forty-three members of the Society. Seventeen papers were read in the three sessions and a most enjoyable time was spent at the dinner on Friday evening at the Quadrangle Club. S.

Professor J. McKeen Cattell, Columbia University, lectured, on January 22, before the Senate of the University of Illinois, on the subject of University Administration. Professor Cattell holds the view that administrative officers, including the President of the University, should not receive any larger salaries than the most competent professors are paid. He would also put the emphasis on good men rather than on fine buildings and grounds. M.

In a series of articles under the title "Twelve Major Prophets of Today," now appearing in *The Independent*, Henri Poincaré, the eminent French mathematician is the subject of the third paper. The author, Dr. Edwin E. Slosson, pays the following tribute to one of the *Monthly's* contributors: "In this country Poincaré has become known largely through the efforts of Professor George Bruce Halsted of the State Normal School of Greeley, Colorado, who has translated his philosiphical works and has for many years been indefatigable in spreading the new gospel of the non-euclidean geometry. S.

The sixty-third meeting of the American Association for the Advancement of Science was held at Washington, D. C., December 27 to December 30, and was attended by about 2500 members of the Association and affiliated societies. At this meeting Professor Moore gave his retiring address as chairman of Section A. His subject was, "On the foundation of the theory of linear integral equations. Professor VanVleck, University of Wisconsin, was elected chairman of Section A for the next meeting, which is to be held at Cleveland, Ohio, during the week in which January first, 1913, falls. M.

The United States Bureau of Education has recently issued Bulletins Numbers 13 and 16 for 1911, the former containing the Report of the American Committees I and II, on Mathematics in the Elementary Schools of the United States; and the latter containing the Report of the American Committees III and IV, on Mathematics in the Public and Private Secondary Schools. These Reports are prepared under the direction of the American Commissioners of the International Commission on the Teaching of Mathematics. They may be secured gratis by addressing the United States Commissioner of Education at Washington. S.

At the annual meeting of the Central Association of Science and Mathematics Teachers held at Lewis Institute in Chicago, there were presented at the Mathematics section two important reports of committees: (1) on Results of Mathematical Teaching, Ascertained by Scientific Spirit in the Daily Work and by Scientific Methods of Testing Efficiency; and (2) on Uniform Notation for Algebra. The first report was presented by Professor C. E. Comstock of Bradley Polytechnic Institute, Peoria, Illinois, and and was discussed by Mr. Charles Otterman, of Cincinnati, Ohio, and the second was presented by Mr. L. P. Jocelyn of Ann Arbor, Michigan, High School, and was discussed by Professor E. R. Hedrick, of the University of Missouri, and by Professor George R. Twin, of Ohio State University. An interesting paper also was presented by Mr. K. G. Smith, of the University of Wisconsin on, "The Applications of Mathematics to Problems of the Shop." The following officers were elected for the year 1912; Chairman, Ira S. Condit, State Teachers College, Cedar Falls, Iowa; Vice-Chairman, Charles W. Newhall, Shattuck School, Faribault, Minnesota; Secretary, Miss Marie Gugle, Central High School, Toledo, Ohio. S.

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SOME MECHANICAL DEVICES TO GENERATE CERTAIN SYSTEMS OF CURVES.

By ARNOLD EMCH, University of Illinois.

1. The construction of curves and surfaces by mechanical means, especially by linkages, has been the object of a number of investigations. Some of the most important results along these lines are due to Kempe* and Koenigs.† The first proved that all algebraic curves in a plane may be generated by link-motions. Koenigs found the same to be true for algebraic curves in space. In a further generalization the writer has shown‡ that every algebraic transformation between any number of variables, and as a consequence between any number of complex variables, may be realized by linkages. In another place§ I have, in particular, described linkages for continuous groups of collineation in a plane. These imply the description of conics.

Based upon an articulated regulus‖ it is possible to perform by link-motions certain transformations of a plane into particular quadrics and quartic surfaces and to describe twisted curves of higher order.¶

There are, however, a great number of other mechanical and physical devices besides link-motions by which various systems of curves, for example Lissajous curves in acoustics, may be generated. The following simple examples, in which again ruled quadrics appear, may be of interest in this respect.

2. Consider a variable string model of a regulus in which strings of equal length are used. One end of each string is attached to a point A' of the directrix X', while the other extremity of the string is allowed to slip freely through the corresponding point A of a second directrix X and is kept

* "How to draw a straight line," 1877.

† *Lecons de Cinematique*, 1897, pp. 302-307.

‡ *Transactions of the American Mathematical Society*, Vol. III (1902), pp. 493-498.

§ *Introduction to Projective Geometry and Its Applications*, 1905, pp. 242-260.

‖ A regulus is a ruled quadric.

¶ *Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen*, Solothurn, 1906. *Archives der sciences physiques et naturelles*, Vol. 24 (1907), p. 368.